EXERCISE – IV HINTS & SOLUTIONS

Sol.1 Equation of normal to the ellipse

$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$ .....(i)

so point P = $(a\cos\theta, b\sin\theta)$

point G = $\left(\dfrac{a^2-b^2}{a}\cos\theta, 0\right)$

Let point Q = (h, k)

so $ah \sec\theta - bk \operatorname{cosec}\theta = a^2 - b^2$.....(ii)

& Given PQ = GP

$\Rightarrow (h - a\cos\theta)^2 + (k - b\sin\theta)^2$

$= \left(a\cos\theta - \dfrac{a^2-b^2}{a}\cos\theta\right)^2 + (b\sin\theta)^2$.....(iii)

using (ii) & (iii), locus of Q is ellipse

Sol.2 Point P = $(a\cos\theta, b\sin\theta)$

& Q = $(a\cos\theta, a\sin\theta)$

equation of tangent to ellipse at 'P'

$bx\cos\theta + ay\sin\theta = ab$

point T = $(a\sec\theta, 0)$

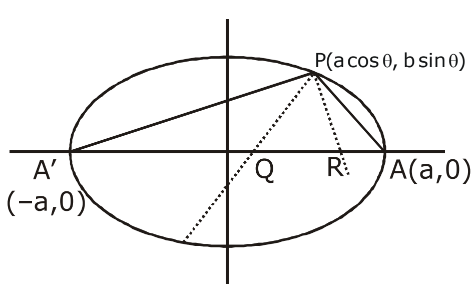
equation of line joining Q & T

$y = \dfrac{a\sin\theta}{a\cos\theta - a\sec\theta}(x - a\sec\theta)$

which is tangent to the circle $x^2 + y^2 = a^2$

Sol.3 $m_{PA} = \dfrac{b\sin\theta}{a\cos\theta - a}$

$m_{PQ} = \dfrac{a - a\cos\theta}{b\sin\theta}$



Eq^n of PQ:

$y - b\sin\theta \; \dfrac{a - a\cos\theta}{b\sin\theta}(x - a\cos\theta)$

For point Q: but y = 0

$Q\left[a\cos\theta - \dfrac{b^2}{a}(1+\cos\theta), 0\right]$

$m_{PA'} = \dfrac{b\sin\theta}{a\cos\theta - a}$

$m_{PR} = \dfrac{-(a\cos\theta + a)}{b\sin\theta}\;(x - a\cos\theta)$

Equation of PR

$y - b\sin\theta = -\dfrac{(a\cos\theta + a)}{b\sin\theta}(x - a\cos\theta)$

For point R put y = 0

$R\left[a\cos\theta + \dfrac{b^2}{a}(1-\cos\theta), 0\right]$

$\ell(QR) = \dfrac{2b^2}{a}$

Sol.4 Equation of parabola,

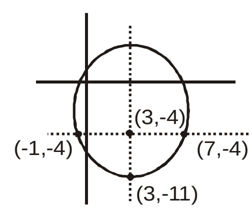
$(x-3)^2 = k(y+11)$

which is passing through

$(7, -4) \Rightarrow k = 16/7$

$\therefore 16y = 7(x-3)^2 - 176$

$\Rightarrow \quad a + h + k = 186$

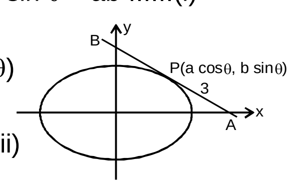


Sol.5 Equation of tangent at 'P'

$bx\cos\theta + ay\sin\theta = ab$ ......(i)

point A = $(a\sec\theta, 0)$

point B = $(0, b\operatorname{cosec}\theta)$



Now $a\cos\theta = \dfrac{a\sec\theta}{4}$ ....(ii)

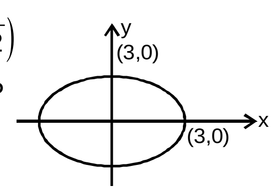
& $b\sin\theta = \dfrac{3b\operatorname{cosec}\theta}{4}$ ....(iii)

using (2) & (3) to get equation of tangent

Sol.6 Point P = $\left(5/\sqrt{2}, 3/\sqrt{2}\right)$

equation of normal at P

$5x - 3y = 8\sqrt{2}$ .....(i)



point A = $\left(\dfrac{8\sqrt{2}}{5}, 0\right)$ & B = $\left(0, \dfrac{-8\sqrt{2}}{3}\right)$.

Tangent at P : $3x + 5y = 15\sqrt{2}$ .....(ii)

Point T = $\left(5\sqrt{2}, 0\right)$ check the options.

Sol.7 Equation of tangent at $(2\cos\theta, \sin\theta)$

$x\cos\theta + 2y\sin\theta = 2$ ....(1)

$x^2 + 2y^2 + 6$ ....(2)

Let a point is (x_1, y_1) on ellipse (2)

at which tangents intersects each other O then equation of COC of (x_1, y_1) is

$xx_1 + 2yy_1 - 6 = 0$ ....(3)

equation (1) & (3) are same, so on companing

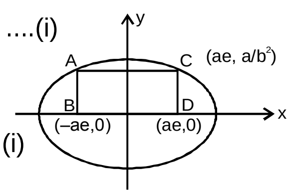
$x_1/3 = \cos\theta$ & $y_1/3 = \sin\theta \Rightarrow x_1^2 + y_1^2 = 9$

locus of $(x_1, y_1) \Rightarrow x^2 + y^2 = 9$

which is the director circle of ellipse (2), hence tangents cuts each other at 90º.

Sol.8 $\pi ab = 200\pi$ $ab = 200$ ....(i)

Area of rectangle



$(2ae) \times \dfrac{a}{b^2} = 200$ ....(i)

use (i) & (ii) to get a & b.

Sol.9 **(a)** c_1 ; $y^2 = 4x$

$2yy_1 = 4$

$m_1 = y_1 = \dfrac{4}{2y} = \dfrac{2}{y}$

c_2 ; $2x^2 + y^2 = 6$

$4x + 2yy^1 = 0$

$m_2 = y_1 = -\dfrac{2x}{y}$

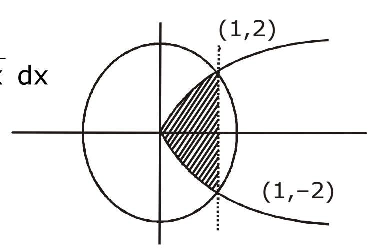
$m_1m = -\dfrac{4x}{y^2} = -\dfrac{4x}{4x} = -1$

curves are orthagonal.

(b)

Area = $\int_0^1 \sqrt{4x}\, dx$

= 8/3



(c)

Tangent at P

$y + x = 3$

$\Rightarrow T(3, 0)$

Normal at P

$x - y = -1$

$\Rightarrow G(-1, 0)$

Area = $\dfrac{1}{2} \times 2 \times 4 = 4$

Sol.10 Equation of normal with slope 1

$$y = x - \frac{(a^2 - b^2)}{\sqrt{a^2 + b^2}} \quad .....(i)$$

point P = $\left(\dfrac{(a^2 - b^2)}{\sqrt{a^2 + b^2}}, 0\right)$; Q = $\left(0, -\dfrac{(a^2 - b^2)}{\sqrt{a^2 + b^2}}\right)$

C = (0, 0)

$$\text{Area of } \Delta CPQ = \frac{1}{2}\begin{vmatrix} \dfrac{a^2 - b^2}{\sqrt{a^2 + b^2}} & 0 & 1 \\ 0 & \dfrac{-(a^2 - b^2)}{\sqrt{a^2 + b^2}} & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{(a^2 - b^2)^2}{2(a^2 + b^2)} \text{ sq. units.}$$

Sol.11 Equation of tangent at $\left(\dfrac{a^2}{\sqrt{a^2 - b^2}}, \sqrt{a^2 + b^2}\right)$

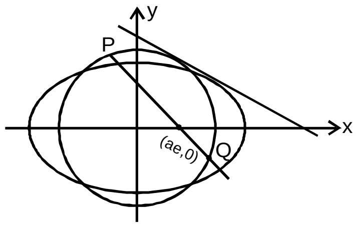
$$\frac{x.a^2}{\sqrt{a^2 - b^2}.a^2} + \frac{y.\sqrt{a^2 + b^2}}{b^2} = 1$$

$\Rightarrow b^2x + y\sqrt{a^4 - b^4} = b^2\sqrt{a^2 - b^2}$ .....(i)

Intercept on x = ae

Sol.12 Equation of tangent to ellipse

$bx \cos \theta + ay \sin \theta = ab$ ....(1)



If this is also tangent to the circle then

$$\left|\frac{-ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}\right| = r^2 \quad ....(2)$$

so equation of chond parallel to the tangent :

$$y = -\frac{\sqrt{r^2-b^2}}{a^2-r^2}x + \left(\sqrt{\frac{r^2-b^2}{a^2-r^2}}\right).ae \quad ....(3)$$

(possing through (ae, 0))

Length of perpendicular from (0, 0) on (3)

$$\ell = \sqrt{r^2-b^2}$$

Sol.13 For the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$

focii are (–4, 0) & (4, 0)

As we know that a ray emanating from one focus, after first reflection passes through the other focus.

so point on the ellipse are A(12/5, 3) & B(–12/5, 3)

so Reflecting ray will be equation of line joining (4, 0) & A and (4, 0) & B.

Sol.14 Tangent at $P(x_1, y_1)$

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Let 5 be (ae, 0) then $P = \left|\frac{\frac{ae\,x_1}{a_2} - 1}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}}}\right|$

$$\Rightarrow \frac{1}{p^2} = \frac{x_1^2/a^2 + y_1^2 a^2/b^4}{(a-ex_1)^2}$$

$$\Rightarrow \frac{b^2}{p^2} = \frac{\left(\frac{b^2x_1^2}{a^2} + a^2\left(1-\frac{x_1^2}{a^2}\right)\right)}{(a-ex_1)^2}$$

$$\Rightarrow \frac{b^2}{p^2} = \frac{(a^2-(1-b^2/a^2)\,x_1^2)}{(a-ex_1)^2}$$

$$\Rightarrow \frac{b^2}{p^2} = \frac{a^2-e^2x_1^2}{(a-ex_1)^2} = \frac{a+ex_1}{a-ex_1} = \frac{2a}{\ell(SP)} - 1$$

$\because$ (ℓSP = focal distance)

Sol.15 Point P = $\left(\sqrt{2}, 1/\sqrt{2}\right)$

shifting the ellipse by leting the origin at

$\left(\sqrt{2}, 1/\sqrt{2}\right)$ $\left(x+\sqrt{2}\right)^2 + 4\left(y+1/\sqrt{2}\right)^2 = 4$

$\Rightarrow x^2 + 4y^2 + 2\sqrt{2}\,x + 8\sqrt{2}\,y = 0$ ....(1)

Let the line AB $\ell x + my = 1$ ....(2)

Homozining (1) with (2) & as the angle between the chords is 90º so coff. of x^2 + coff. of $y^2 = 0$

$\Rightarrow 2\sqrt{2}\,\ell + 4\sqrt{2}\,m = -5$ ....(3)

using (2) & (3) $\left(\frac{-5}{2\sqrt{2}}x - 1\right) + m(y-2x) = 0$ ....(4)

which shows a family of line & passes through a fixed point which is point of intersection of two line A.

$$\Rightarrow x = -\frac{2\sqrt{2}}{5} \quad \& \quad y = \frac{4\sqrt{2}}{5}$$

again $x = -\frac{2\sqrt{2}}{5} - \sqrt{2} = -\frac{3\sqrt{2}}{5}$ & $y = \frac{3\sqrt{2}}{10}$

$$a^2 + b^2 = \frac{9}{10} \Rightarrow a + b = 19$$